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THE COMPUTATIONS OF SOME SCHUR INDICES

BY
WALTER FEIT[†]

ABSTRACT

Let χ be an irreducible character of a finite group G. Let $p = \infty$ or a prime. Let $m_p(\chi)$ denote the Schur index of χ over $\mathbf{Q}_p$, the completion of $\mathbf{Q}$ at p. It is shown that if x is a p'-element of G such that $\chi_u(x) \in \mathbf{Q}_p(\chi)$ for all irreducible characters χ_u of G then $m_p(\chi) \mid \chi(x)$. This result provides an effective tool in computing Schur indices of characters of G from a knowledge of the character table of G. For instance, one can read off Benard's Theorem which states that every irreducible character of the Weyl groups $W(E_n)$, $n = 6, 7, 8$ is afforded by a rational representation. Several other applications are given including a complete list of all local Schur indices of all irreducible characters of all sporadic simple groups and their covering groups (there is still an open question concerning one character of the double cover of Suz).

§1. Introduction

For p a prime or $p = \infty$ let $\mathbf{Q}_p$ denote the completion of $\mathbf{Q}$ at p. If G is a finite group and χ is an irreducible character of G let $m_p(\chi)$ denote the Schur index of χ with respect to $\mathbf{Q}_p$ and let $m(\chi)$ denote the Schur index of χ with respect to $\mathbf{Q}$.

THEOREM A. *Let G be a finite group and let χ be an irreducible character of G. Let p be a prime and let x be a p'-element of G. Assume that $\chi_u(x) \in \mathbf{Q}_p(\chi)$ for every irreducible character χ_u of G. Then $m_p(\chi) \mid \chi(x)$ in the ring of algebraic integers.*

Theorem A is a consequence of a slightly more general result proved in Section 3. The proof is quite simple. Section 3 also contains various corollaries of Theorem A.

The utility of Theorem A and the other results in Section 3 lies in the fact that the relevant information can be derived from the character table of G, together

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with a knowledge of the order of the elements in each class. This is something that can be read off with relatively little effort. This together with other known results about Schur indices make it possible to show that $m(\chi)=1$ for many irreducible characters of many groups. For instance, Theorem A applied to the character tables of the Weyl groups of type E_6, E_7, E_8 immediately yields that all irreducible characters of these groups have Schur index 1. This gives an alternative proof of Benard's Theorem [1] which asserts that all the irreducible characters of these groups are afforded by rational representations.

Of course one can never deduce directly from Theorem A that $m(\chi)\neq 1$ for some irreducible character χ. Nevertheless Theorem A can be very helpful when used in conjunction with other known properties of the Schur index. Several of these are listed without proof in Section 2. An example is worked out in detail in Section 4.

The combination of all these techniques is quite effective as is illustrated in the rest of this paper. Section 5 contains the computation of the local Schur indices of all irreducible characters of all simple groups of order at most 10^6. Section 6 consists of an alternative proof of a result of G. J. Janusz [15] and contains the local Schur indices of all irreducible characters of the groups $SL_2(q)$. Section 7 contains the computations of the local Schur indices of all the irreducible characters of all the sporadic groups and all of their central extensions. The results are summarized in Section 8.

While Theorem A is surprisingly effective, it is not by itself, even in conjunction with Theorem 2.15, sufficient to show that $m_p(\chi)=1$ in all cases when this is so. For example, let ψ be the irreducible character of $Sp_4(4)$ with $\psi(1)=18$. Theorem A shows that $m_p(\psi)=1$ for $p\neq 2,5$ and $m_p(\psi)\leqq 2$ for $p=2,5$. However, as is shown in Section 5, $m_p(\psi)=1$ for all p. It is also true in this case that if ψ_1, ψ_2 are rational valued irreducible characters of $Sp_4(4)$ distinct from ψ then $(\psi_1\psi_2,\psi)$ is even. Thus another simple method is ineffective for this character.

In the last resort Schur indices can be computed by studying characters of quasi-elementary subgroups and applying Theorem 2.2. If $p\neq 2$, the Schur index $m_p(\chi)$ can be computed for an irreducible character χ of a quasi-elementary group by using, for instance, Theorem 2.12. Unfortunately, it is frequently difficult to apply Theorem 2.2 as this requires precise information about the structure of quasi-elementary subgroups and their characters. In this paper Theorem 2.2 is only appealed to when it seems to be unavoidable.

Theorem 2.7 due to Frobenius and Schur gives an effective way of computing $m_\infty(\chi)$. There is no known analogue of this for $p\neq\infty$. Theorem 2.12 due to

Benard is effective in case χ is in a p-block with a cyclic defect group. It would be desirable to find a suitable generalization which works in all cases. Theorem A and Theorem 2.12 differ from most other results about Schur indices in that they require considerations of the columns of the character table, i.e., they do not apply to each irreducible character in isolation but need some information about all the characters of the group.

Over the years I have received character tables from many individuals; these are acknowledged in the appropriate places below. However, I especially want to thank the following people.

D. C. Hunt who computed the rational character table of BM and its cover, and who also computed the character table of Fi_{24}.

J. Neubüser who supplied me with character tables of many groups, including finally a complete set of these for all the simple sporadic groups.

E. Cleuvers who performed several computations in answer to questions which I had raised.

§2. Properties of the Schur index

This section contains a list of properties of the Schur index. Definitions and elementary properties can, for instance, be found in [7]. The following notation will be used:

G is a finite group and χ will always denote an irreducible character of G.

K is a field of characteristic 0 and $m_K(\chi)$ is the Schur index of χ with respect to K.

$m(\chi) = m_{\mathbf{Q}}(\chi)$.

THEOREM 2.1. *Suppose that the character θ of G is afforded by a $K[G]$ module. Then $m_K(\chi) \mid (\chi, \theta)$.*

Let p be a prime and let $H = AP$ where $A = \langle x \rangle$ is a cyclic p'-group with $A \lhd H$ and P is a p-group. Let ε be a primitive $|A|$th root of 1. Then $H = AP$ is *K-elementary with respect to p* if x^i, x^j are conjugate in H only in case ε^i, ε^j are conjugate under the action of the Galois group of $K(\varepsilon)$ over K.

The next result is due to Brauer [6] and was later rediscovered by Witt [23]. A proof can be found in [7].

THEOREM 2.2. *Let ε be a primitive $|G|$th root of 1 and let p be a prime. Let L be a field such that $K(\chi) \subseteq L \subseteq K(\varepsilon)$ and $[K(\varepsilon):L]$ is the p-part of $[K(\varepsilon):K(\chi)]$. Then there exists a subgroup H of G which is K-elementary with respect to p, and an irreducible character ζ of H such that $\mathbf{Q}(\zeta) \subseteq L$, $p \nmid (\zeta^G, \chi)$*

and $m_K(\chi)$ *and* $m_K(\zeta)$ *are divisible by the same power of* p.

In view of Theorem 2.2 the calculation of $m_K(\chi)$ can be reduced to calculating Schur indices of irreducible characters ζ of K-elementary groups and studying the corresponding induced characters ζ^G. It may however be rather complicated to survey all K-elementary subgroups of G and the remaining results in this section and the next are aimed at avoiding the direct use of Theorem 2.2 as much as possible. Of course many of these results are based on Theorem 2.2.

THEOREM 2.3 (Benard and Schacher [5]). *The field* $\mathbf{Q}(\chi)$ *contains a primitive* $m(\chi)$*-th root of* 1.

This result in particular implies

COROLLARY 2.4 (Brauer and Speiser). *If* χ *is real valued then* $m(\chi)\mid 2$.

Define

$$\nu(\chi)=\frac{1}{|G|}\sum_x \chi(x^2).$$

For $x \in G$ let $1+t(x)$ be the number of elements y in G with $y^2 = x$. The orthogonality relations imply that

$$1+t(x)=\sum_\chi \nu(\chi)\chi(x). \tag{2.5}$$

In particular

$$1+t=\sum_\chi \nu(\chi)\chi(1), \tag{2.6}$$

where $t = t(1)$ is the number of involutions in G. For a proof of the next result see, for instance, [7].

THEOREM 2.7 (Frobenius and Schur). (i) *If* $\chi \neq \bar{\chi}$ *then* $\nu(\chi)=0$.

(ii) *Let* $\mathbf{R}$ *denote the field of real numbers. If* $\chi=\bar{\chi}$, *one of the following occurs*:

(a) $m_{\mathbf{R}}(\chi)=1$ *and* $\nu(\chi)=1$.

(b) $m_{\mathbf{R}}(\chi)=2$ *and* $\nu(\chi)=-1$.

Theorem 2.7 gives an effective way of computing $m_{\mathbf{R}}(\chi)$. Together with (2.6) it yields the following useful consequence.

COROLLARY 2.8. *If* $\Sigma_{\chi=\bar{\chi}}\chi(1)=1+t$ *where* t *is the number of involutions in* G *then* $m_{\mathbf{R}}(\chi)=1$ *for every irreducible character* χ *of* G.

This next result is a weak form of a theorem of Benard [2].

THEOREM 2.9. *Let p_1, p_2 be prime divisors in $\mathbf{Q}(\chi)$ of the rational prime p. Let K_i be the completion of $\mathbf{Q}(\chi)$ at p_i for $i = 1,2$. Then $m_{K_1}(\chi) = m_{K_2}(\chi)$.*

An analogous result holds for Archimedean completions by Theorem 2.7. Thus $m_{\mathbf{Q}_p}(\chi) = m_{\mathbf{Q}_p}(\chi')$ if χ and χ' are algebraically conjugate, for $p = \infty$ or p any prime. Hence the following are well defined:

$$m_\infty(\chi) = m_{\mathbf{R}}(\chi), \qquad \text{where } \mathbf{R} \text{ is the field of real numbers,}$$

$$m_p(\chi) = m_{\mathbf{Q}_p}(\chi) \qquad \text{for any rational prime } p.$$

This notation will be used throughout the rest of this paper.

The next result is due to Brauer. For a proof see e.g. [8] theorem IV.9.3.

THEOREM 2.10. *Let $\{\chi_u\}$ be the set of all irreducible characters of G. Let p be a prime and let $\{\varphi_i\}$ be the set of all irreducible Brauer characters of G. Let $D = (d_{ui})$ be the decomposition matrix. Then*

$$m_p(\chi_u) \mid d_{ui}\,[\mathbf{Q}_p(\chi_u, \varphi_i) : \mathbf{Q}_p(\chi_u)]$$

for all u, i. In particular $m_p(\chi) = 1$ if χ is irreducible as a Brauer character.

COROLLARY 2.11. *If $p \nmid |G|$ then $m_p(\chi) = 1$ for every irreducible character χ of G. Hence, in particular, $m_p(\chi) \neq 1$ for only a finite number of primes.*

The next result is due to Benard [3]. An alternative proof can be found in [8], chapter VII, section 13.

THEOREM 2.12. *Let p be a prime and let B be a block with a cyclic defect group. Then $F = \mathbf{Q}_p(\varphi)$ is the same for every irreducible Brauer character φ in B. Furthermore,*

$$m_p(\chi) = [F(\chi) : \mathbf{Q}_p(\chi)]$$

for every irreducible character χ in B.

(i) *If B contains exceptional characters and χ is nonexceptional then $\mathbf{Q}_p(\chi) = F$ and so $m_p(\chi) = 1$.*

(ii) *If B does not contain exceptional characters then $\mathbf{Q}_p(\chi) = F$ for every irreducible character in B with at most one exception. Thus $m_p(\chi) = 1$ for all but at most one irreducible character χ in B.*

COROLLARY 2.13. *Let p be a prime and let B be a block with a cyclic defect group. If B contains an irreducible Brauer character φ with $\mathbf{Q}_p(\varphi) = \mathbf{Q}_p$ then $m_p(\chi) = 1$ for every irreducible character χ in B. In particular this is the case if B is the principal block.*

The remaining results in this section follow from the structure of division algebras over algebraic number fields and their completions.

Let P denote the set of all rational primes.

Let $\Delta(\chi)$ be the simple algebra component of $\mathbf{Q}[G]$ corresponding to χ. Thus $\mathbf{Q}(\chi)$ is the center of $\Delta(\chi)$.

Let $V(\chi)$ denote the set of all (equivalence classes of) valuations of $\mathbf{Q}(\chi)$.

For $v \in V(\chi)$ let $\Delta(\chi)_v = \Delta(\chi) \otimes_{\mathbf{Q}(\chi)} \mathbf{Q}(\chi)_v$, where $\mathbf{Q}(\chi)_v$ is the completion of $\mathbf{Q}(\chi)$ at v.

Then $m(\chi)$ is the Schur index of $\Delta(\chi)$ and if $v \in V(\chi)$ such that $v \mid p$ for $p \in P \cup \{\infty\}$, then $m_p(\chi)$ is the Schur index of $\Delta(\chi)_v$.

THEOREM 2.14. *$\Delta(\chi)$ is determined up to isomorphism by the set of $\Delta(\chi)_v$, $v \in V(\chi)$ and $m(\chi)$ is the least common multiple of all $m_p(\chi)$, $p \in P \cup \{\infty\}$. Furthermore if $m(\chi) = 2$ then $\Delta(\chi)$ is determined up to isomorphism by the set of all $m_p(\chi)$, $p \in P \cup \{\infty\}$.*

THEOREM 2.15. *Let q be a prime which divides $m(\chi)$ and let q^c be the exact power of q which divides $m(\chi)$. Let $V_0 = \{v \mid v \in V(\chi),\ q^c \mid m_{\mathbf{Q}(\chi)_v}(\chi)\}$. Then $|V_0| > 1$. Furthermore if $q^c = 2$ then $|V_0|$ is even.*

THEOREM 2.16. *Let K be a finite extension of $\mathbf{Q}_p$. Then*

$$m_p(\chi) = m_K(\chi)(m_p(\chi), [K(\chi) : \mathbf{Q}_p(\chi)]).$$

§3. Proof of Theorem A

Let G be a finite group and let p be a prime. Let $\{\chi_u\}$ be the set of all irreducible characters of G and let $\{\varphi_i\}$ be the set of all irreducible Brauer characters of G. Let (d_{ui}) be the decomposition matrix.

THEOREM 3.1. *Let K be a finite extension field of $\mathbf{Q}_p$. Let x be a p'-element in G. Let $\chi = \chi_u$ be an irreducible character of G. Assume that $\varphi_i(x) \in K(\chi)$ for all i with $d_{ui} \neq 0$. Then $m_K(\chi) \mid \chi(x)$ in the ring of algebraic integers.*

PROOF. Since $m_K(\chi) \mid m_p(\chi)$ it suffices to prove the result in case $K = \mathbf{Q}_p(\chi)$.

Let H be the Galois group of a splitting field of $K[G]$ over K. For each i let $\hat{\varphi}_i$ be the sum of all the distinct conjugates of φ_i under the action of H. Thus $\chi(y) = \sum d_{ui}\hat{\varphi}_i(y)$ for all p'-elements y in G, where i ranges over a suitable index set. Then $\hat{\varphi}_i$ is the sum of $[K(\varphi_i) : K]$ algebraically conjugate Brauer characters. Thus by assumption

$$\chi(x) = \sum d_{ui}[K(\varphi_i) : K]\varphi_i(x).$$

Hence Theorem 2.10 implies the result.

The following result clearly implies Theorem A.

COROLLARY 3.2. *Let x be a p'-element in G. Let B be the block containing χ. Assume that $\chi_v(x) \in \mathbf{Q}_p(\chi)$ for all χ_v in B. Then $m_p(\chi) \mid \chi(x)$ in the ring of algebraic integers.*

PROOF. An irreducible Brauer character in B is an integral linear combination of irreducible characters in B as functions on p'-elements. Thus the result follows from Theorem 3.1.

For the remaining corollaries in this section let $\chi = \chi_u$ be an irreducible character of G. Let n be a natural number and let x be an element of G with $x^n = 1$.

COROLLARY 3.3. *Suppose that $\chi_v(x) \in \mathbf{Q}_p(\chi)$ for all v and all primes p with $p \nmid n$. Then the following hold:*

(i) *$m_p(\chi) \mid \chi(x)$ for all primes p with $p \nmid n$.*

(ii) *If n is a power of the prime q then $m_p(\chi) \mid \chi(x)$ for all $p \neq q, \infty$.*

The following two results arise frequently in applications.

COROLLARY 3.4. *Suppose that $\chi_v(x) \in \mathbf{Q}_p(\chi)$ for all v and all primes p with $p \nmid n$. Assume that $m(\chi) \mid 2$ and $2 \nmid \chi(x)$. Then $m_p(\chi) = 1$ for all primes p with $p \nmid n$.*

COROLLARY 3.5. *Suppose that n is a power of the prime q. Assume that x is a rational element (i.e., $\chi_u(x) \in \mathbf{Q}$ for all v). Assume further that $\chi(x)$ is odd and $\mathbf{Q}(\chi) = \mathbf{Q}$. Then $m_p(\chi) = 1$ for $p \neq q, \infty$ and*

$$m_q(\chi) = m_\infty(\chi) = m(\chi) = 1 \text{ or } 2.$$

PROOF. By Corollary 3.4 $m_p(\chi) = 1$ for $p \neq q, \infty$. By Corollary 2.4 $m(\chi) \mid 2$. The result now follows from Theorems 2.14 and 2.15.

§4. An example

In this section we will give all the details needed to compute the Schur indices of all the faithful irreducible characters of the double cover $G = 2HJ$ of the simple group $\bar{G} \simeq HJ$. The table of faithful characters is reproduced here from [17]. See Table I. We will also sketch the computation of the Schur indices of the irreducible characters of $\bar{G} \simeq HJ$.

The simple group $\bar{G}$ has 2835 involutions. The group G has 631 involutions. Thus

$$\sum_{u=1}^{17} \nu(\chi_u)\chi_u(1) = 632 - 2836 = -2204.$$

By inspection χ_4 and χ_5 are the only non-real valued characters in Table I and

$$\sum_{\substack{u=1\\ u\neq 4,5}}^{17} \chi_u(1) = 2204.$$

It follows that $m_\infty(\chi_u) = 2$ for $1 \leqq u \leqq 17$, $u \neq 4, 5$.

By Theorem 2.3 $m(\chi_u)\,|\,2$ for $1 \leqq u \leqq 17$.

By Corollary 2.11 $m_p(\chi_u) = 1$ for all u and all primes $p \neq 2, 3, 5, 7$.

By Theorem 2.10 $m_7(\chi_u) = 1$ for χ_u of 7-defect 0. Suppose that χ_u has 7-defect 1. By Theorem A $m_p(\chi_u) = 1$ for $p \neq 7, \infty$. Let F be the quadratic unramified extension of $\mathbf{Q}_7$. Then $\mathbf{Q}_7(\chi_u) = F$ for $u = 1, 2, 4, 5, 8, 9$ and $\mathbf{Q}_7(\chi_{13}) = \mathbf{Q}_7$. Thus by Theorem 2.12 $m_7(\chi_u) = 1$ for $u = 1, 2, 4, 5, 8, 9$ and $m_7(\chi_{13}) = 2$. We conclude from Theorem 2.15 that the following hold:

$$m(\chi_u) = m_\infty(\chi_u) = 2 \quad \text{for } u = 1, 2, 8, 9. \quad \text{(There are 2 Archimedean valuations.)}$$

$$m(\chi_4) = m(\chi_5) = 1.$$

$$m(\chi_{13}) = m_\infty(\chi_{13}) = m_7(\chi_{13}) = 2.$$

All other local Schur indices are 1 for these characters.

By Theorem A $m_p(\chi_u) = 1$ for $p \neq 3$ and $u = 10, 11, 12, 14$. Since $\mathbf{Q}(\chi_{10}) = \mathbf{Q}(\chi_{14}) = \mathbf{Q}$ it follows from Theorem 2.15 that

$$m(\chi_u) = m_3(\chi_u) = m_\infty(\chi_u) = 2 \qquad \text{for } u = 10, 14,$$

and $m_p(\chi_u) = 1$ for $u = 10, 14$ and $p \neq 3, \infty$.

The characters χ_{11}, χ_{12}, χ_{14} form a 3-block of defect 1. Since $\mathbf{Q}_3(\chi_{11}) = \mathbf{Q}_3(\chi_{12})$ is a quadratic extension of $\mathbf{Q}_3$, Theorem 2.12 implies that $m_3(\chi_{11}) = m_3(\chi_{12}) = 1$ (and $m_3(\chi_{14}) = 2$ confirming an earlier result). Thus

$$m(\chi_u) = m_\infty(\chi_u) = 2 \qquad \text{for } u = 11, 12.$$

Furthermore $m_p(\chi_u) = 1$ for $u = 11, 12$, $p \neq \infty$. (There are 2 Archimedean valuations.)

Let $\bar{x}$ be a 5-element with $|C(\bar{x})| = 50$. Then $\chi_u(x) = (-3 \pm \sqrt{5})/2$ for $u = 6, 7$. Thus by Theorem A $m_p(\chi_u) = 1$ for $u = 6, 7$ and $p \neq 5, \infty$. There are 2 Archimedean valuations in $\mathbf{Q}(\chi_6) = \mathbf{Q}(\chi_7)$. There is only one prime divisor of 5 in $\mathbf{Q}(\sqrt{5})$. Hence Theorems 2.9 and 2.15 imply that

$$m(\chi_u) = m_\infty(\chi_u) = 2 \qquad \text{for } u = 6, 7$$

and $m_p(\chi_u) = 1$ for $u = 6, 7$, $p \neq \infty$.

This leaves the 4 rational valued characters χ_3, χ_{15}, χ_{16}, χ_{17}. Unfortunately, Theorem A cannot be applied with x of order 5 since $[\mathbf{Q}_p(\sqrt{5}):\mathbf{Q}_p]=2$ for $p=2,3,5$. Before looking further at these we will consider the characters of $\bar{G}\simeq HJ$. A character table can be found in [19]. Benard [4] has found all Schur indices for this group by using Theorem 2.2 and induce-restrict tables for certain subgroups. We will derive his results here more directly.

By counting involutions it is straightforward to verify that $m_\infty(\psi)=1$ for every irreducible character ψ of $\bar{G}$. By Theorem 2.3 $m(\psi)\,|\,2$ for all such ψ. It is easy to verify by using Theorem A and the results of Section 2 other than Theorem 2.2 that $m(\psi)=1$ for every irreducible character except ψ_{21} where $\psi_{21}(1)=336$. It should be emphasized that this result can be read off directly from the character table and does not involve the consideration of any proper subgroup of $\bar{G}$.

There are irreducible characters ψ_6, ψ_{10}, ψ_{12} of $\bar{G}$ with $\psi_6(1)=36$, $\psi_{10}(1)=90$, $\psi_{12}(1)=160$ and $\mathbf{Q}(\psi_6)=\mathbf{Q}(\psi_{10})=\mathbf{Q}(\psi_{12})=\mathbf{Q}$. Direct computation shows that

$$(\chi_{15},\chi_3\psi_6)=(\chi_3,\chi_{15}\psi_6)=(\chi_3\chi_{15},\psi_6)=1,$$

$$(\chi_{16},\chi_3\psi_{10})=(\chi_3,\chi_{16}\psi_{10})=(\chi_3\chi_{16},\psi_{10})=1,$$

$$(\chi_{17},\chi_3\psi_{12})=(\chi_3,\chi_{17}\psi_{12})=(\chi_3\chi_{17},\psi_{12})=1.$$

By Theorem 2.1 this implies that for all p

$$m_p(\chi_3)=m_p(\chi_{15})=m_p(\chi_{16})=m_p(\chi_{17}). \tag{4.1}$$

At this point it appears to be necessary to consider proper subgroups for G. There exists an element y of order 5 in G such that $\mathbf{C}_{\bar{G}}(\bar{y})=\langle\bar{y}\rangle\times A$, where $A\simeq A_5$. The group $\mathbf{C}_G(y)$ cannot contain a copy of A_5 as $(\chi_1)_{\mathbf{C}_G(y)}$ would not be a character. Thus $\mathbf{C}_G(y)=\langle y\rangle\times\tilde{A}$, where $\tilde{A}\simeq \mathrm{SL}_2(5)$. Let T be a S_2-group of $\tilde{A}$. Then T is a quaternion group of order 8. Let ζ be the unique faithful irreducible character of T. Every faithful character of G vanishes on $T-\mathbf{Z}(T)$. Hence $(\chi_u)_T=\frac{1}{2}\chi_u(1)\zeta$ for $1\leqq u\leqq 17$. In particular, $(\chi_3,\zeta^G)=7$. Since $m_p(\zeta)=1$ for $p\neq 2,\infty$ it follows from Theorem 2.1 that $m_p(\chi_3)=1$ for $p\neq\infty,2$. Thus (4.1) and Theorem 2.15 imply that

$$m_\infty(\chi_u)=m_2(\chi_u)=m(\chi_u)=2 \qquad \text{for } u=3,15,16,17$$

and $m_p(\chi_u)=1$ for $u=3,15,16,17$, $p\neq 2,\infty$.

This completes the computation of Schur indices of all faithful irreducible characters of G. It only remains to consider ψ_{21}.

Direct computation shows that

$$(\chi_3,\psi_{21}\chi_{10})=(\chi_{10},\psi_{21}\chi_3)=(\chi_{10}\chi_3,\psi_{21})=1.$$

If $m_2(\psi_{21})=1$ then $\psi_{21}\chi_{10}$ is afforded by a $\mathbf{Q}_2[G]$ module and so $m_2(\chi_3)=1$ by Theorem 2.1 contrary to what has been proved above. If $m_3(\psi_{21})=1$ then $\psi_{21}\chi_3$ is afforded by a $\mathbf{Q}_3[G]$ module and so $m_3(\chi_{10})=1$ contrary to what has been proved above. Thus by Theorem 2.15

$$m_2(\psi_{21})=m_3(\psi_{21})=m(\psi_{21})=2$$

and $m_p(\psi_{21})=1$ for $p\neq 2,3$.

§5. Simple groups of order less than 10^6

The character tables of all simple groups of order at most 10^6 can be found in [19]. The Schur indices of all these characters are probably known but we will indicate here how most of these can be determined very easily.

By making use of Theorem A and some of the results mentioned in Section 2, and without considering any proper subgroups of G, one can very quickly deduce the following result by simply reading the various character tables.

THEOREM 5.1. *Let G be a simple group of order less than 10^6 and let ψ be an irreducible character of G. Then $m_p(\psi)=1$ for p a prime or $p=\infty$ except possibly in the following cases. (Only one character in each set of algebraically conjugate ones is listed.)*

$G\simeq \mathrm{PSU}_3(3)$	$\psi(1)=6$	$\mathbf{Q}(\psi)=\mathbf{Q}$	$m(\psi)=m_\infty(\psi)=m_3(\psi)=2$
$G\simeq \mathrm{PSU}_3(4)$	$\psi(1)=12$	$\mathbf{Q}(\psi)=\mathbf{Q}$	$m(\psi)=m_\infty(\psi)=m_2(\psi)=2$
$G\simeq \mathrm{PSU}_3(5)$	$\psi(1)=20$	$\mathbf{Q}(\psi)=\mathbf{Q}$	$m(\psi)=m_\infty(\psi)=m_5(\psi)=2$
$G\simeq \mathrm{PSL}_3(5)$	$\psi(1)=124$	$\mathbf{Q}(\psi)=\mathbf{Q}(i)$	$1\leqq m(\psi)=m_5(\psi)\leqq 2$
$G\simeq HJ$	$\psi(1)=336$	$\mathbf{Q}(\psi)=\mathbf{Q}$	$1\leqq m_p(\psi)\leqq 2$ for $p=2,3,5$
$G\simeq \mathrm{Sp}_4(4)$	$\psi(1)=18$	$\mathbf{Q}(\psi)=\mathbf{Q}$	$1\leqq m(\psi)=m_2(\psi)=m_5(\psi)\leqq 2$

The remaining open cases can be settled as follows.

If $G\simeq \mathrm{PSL}_3(5)$ then G has irreducible characters ψ_2, ψ_6 with $\mathbf{Q}(\psi_2)=\mathbf{Q}$, $[\mathbf{Q}(\psi_6):\mathbf{Q}]=3$, $\psi_2(1)=30$, $\psi_6(1)=96$ and $(\psi_2\psi_6,\psi)=1$. Thus by Theorem 2.1 $m(\psi)=1$.

If $G\simeq HJ$ it was shown in Section 4 that $m_p(\psi)=1$ for $p\neq 2,3$ and

$$m(\psi)=m_2(\psi)=m_3(\psi)=2.$$

Suppose that $G \simeq \mathrm{Sp}_4(4)$. Here it appears to be necessary to make use of Theorem 2.2. We will show that if $H = AP$ is $\mathbf{Q}_5$-elementary with respect to 2 then $m_5(\zeta) = 1$ for every irreducible character ζ of H.

Let $H = AP$ with P a 2-group. If $5 \nmid |A|$ then $m_5(\zeta) = 1$ for every irreducible character ζ of H by Corollary 2.11. If $5 \,\big|\, |A|$ then $|A| = 5$ or 15.

If $|A| = 15$ then $\mathbf{C}_G(A) = A$ and $A \subseteq H \subseteq D_3 \times D_5$, where D_n is a dihedral group of order $2n$. It is well known that every irreducible character of every subgroup of $D_3 \times D_5$ has Schur index 1.

Suppose that $|A| = 5$. Then $|\mathbf{N}_G(A) : \mathbf{C}_G(A)| = 2$ and a S_2-group of $\mathbf{C}_G(A)$ has exponent at most 2, since G contains no elements of order 20. Thus a S_2-group of $\mathbf{N}_G(A)$ has exponent at most 4. Hence P has exponent at most 4 and so every irreducible Brauer character of H (with respect to $p = 5$) has values in $\mathbf{Q}_5$. By Corollary 2.13 $m_5(\zeta) = 1$ for every irreducible character ζ of H. By Theorem 2.2 $m_5(\psi) = 1$. Hence $m(\psi) = 1$.

§6. The groups $\mathrm{SL}_2(q)$

G. J. Janusz [15] has found all Schur indices $m_p(\chi)$ for all irreducible characters χ of $G = \mathrm{SL}_2(q)$. In this section we will derive a slightly more precise version of his results in a fairly direct manner. The notation of Table II will be used. A character table of G with q even can be found in [19].

THEOREM 6.1. *Let $G = \mathrm{SL}_2(q)$ and let $\langle z \rangle$ be the center of G. Let $\bar{G} = G/\langle z \rangle \simeq \mathrm{PSL}_2(q)$.*

(I) *Every irreducible character of $\bar{G}$ has Schur index 1. In particular, if q is even then every irreducible character of $G = \bar{G}$ has Schur index 1.*

(II) *Suppose that q is odd. Then $m(\chi) \,|\, 2$ for all faithful irreducible characters χ of G. Furthermore $m_p(\chi) = 1$ for $p = \infty$ or p a prime except in the following cases:*

(i) *$m_\infty(\chi) = 2$ unless $\varepsilon = -1$ and $\chi = \xi_1$ or ξ_2.*

(ii) *Suppose that q is not a rational square. Let $\delta = \pm 1$ and let p be a prime with $p \,|\, q + \delta$ and $p \equiv 3 \pmod 4$. Then the following hold:*

If $\delta = \varepsilon = -1$ and ρ^{2i} is a primitive p^kth root of 1 for some natural numbers k then $m_p(\eta_i) = 2$.

If $\delta = \varepsilon = 1$ and σ^{2j} is a primitive p^kth root of 1 for some natural number k then $m_p(\theta_j) = 2$.

(iii) *If $q \equiv 5\varepsilon \pmod 8$ then*

$$m_2(\eta_i) = 2 \qquad \text{if } \varepsilon = 1 \quad \text{and} \quad \rho^{4i} = 1,$$

$$m_2(\theta_j) = 2 \qquad \text{if } \varepsilon = -1 \quad \text{and} \quad \sigma^{4j} = 1.$$

(iv) *If q is a rational square then*

$$m_r(\zeta_1) = m_r(\zeta_2) = 2.$$

The next result is required for the proof of Theorem 6.1.

LEMMA 6.2. *Suppose that q is odd and χ is an irreducible faithful character of G with $\mathbf{Q}(\chi) = \mathbf{Q}$. Then one of the following occurs:*

(i) $\chi = \eta_i$ *and* $\rho^{6i} = 1$.
(ii) $\chi = \theta_j$ *and* $\sigma^{6j} = 1$.
(iii) $q \equiv 5 \pmod 8$, $\chi = \eta_i$ *with* $\rho^{4i} = 1$.
(iv) $q \equiv 3 \pmod 8$, $\chi = \theta_j$ *with* $\sigma^{4j} = 1$.
(v) *q is a rational square and* $\chi = \zeta_1$ *or* ζ_2.

PROOF. If $\chi = \zeta_i$ or ξ_i the result is clear. Suppose that $\chi = \zeta_i$ or θ_j. Then i, j are odd as χ is faithful.

If τ is a primitive vth root of 1 with $\tau + \tau^{-1} \in \mathbf{Q}$ then $v \mid 6$. Thus either $\chi = \eta_i$ and $\rho^{6i} = 1$ or $\rho^{4i} = 1$, or $\chi = \theta_j$ and $\sigma^{6j} = 1$ or $\sigma^{4j} = 1$. If $\rho^{4i} = 1$ then $q \equiv 5 \pmod 8$ as i is odd. Similarly $q \equiv 3 \pmod 8$ if $\sigma^{4j} = 1$.

PROOF OF THEOREM 6.1. If $q \neq 3$ then $m(\chi) \mid 2$ for all χ by Theorem 2.3. If $q = 3$ then $\chi(1) \mid 2$ for all χ and so again $m(\chi) \mid 2$.

If q is even then every character except one, say Γ, has odd degree and so $m(\chi) = 1$ for $\chi \neq \Gamma$. By Theorem A $m_p(\Gamma) = 1$ for all $p \nmid q+1$ and all $p \nmid q-1$. Hence $m_p(\chi) = 1$ for all $p > 2$ and so $m(\chi) = 1$ by Theorem 2.15. If $q = 2$ then $|G| = 6$ and the result is clear.

From now suppose that q is odd.

Let χ be an irreducible character of G with $\langle z \rangle$ in the kernel of χ. Then χ is a character of $\bar{G} \simeq \mathrm{PSL}_2(q)$. Let D be the image in $\bar{G}$ of $\mathbf{N}_G(\langle b \rangle)$. Then D is a dihedral group of order $q+1$. If $\chi = \eta_i$, θ_j, Γ then $(\chi_D, \lambda) = 1$ for some linear character λ of D with $\lambda^2 = 1$. Hence $(\chi, 1_D^G) = 1$ and so $m(\chi) = 1$ by Theorem 2.1.

If $\chi(1) = \frac{1}{2}(q \pm 1)$ then $\chi(1)$ is odd and so $m(\chi) = 1$. This proves (I).

There is only one involution in G and there are exactly $\frac{1}{2}q(q+\varepsilon)$ involutions in $\bar{G}$. Hence (2.6), Theorem 2.7 and Table II imply that

$$\sum_{\chi = \bar{\chi}} \chi(1) = \tfrac{1}{2}q(q+\varepsilon) - 1 = -\sum_{\chi} \nu(\chi)\chi(1),$$

where χ ranges over all faithful irreducible characters of G. Thus $\nu(\chi) = -1$ and hence by Theorem 2.7 $m_\infty(\chi) = 2$ for every faithful irreducible character χ with $\chi = \bar{\chi}$. This proves (II)(i).

From now on suppose that χ is a faithful irreducible character of G.

Suppose first that $\chi(1)=\frac{1}{2}(q-\delta)$ with $\delta=\pm 1$. Then $2 \not| \frac{1}{2}(1 \pm \sqrt{\varepsilon q})$ for a suitable choice of sign even if $\sqrt{\varepsilon q} \in \mathbf{Q}$. Hence Theorem A implies that $m_l(\chi)=1$ for $l \neq r, \infty$. Since $\mathbf{Q}(\chi)=\mathbf{Q}(\sqrt{\varepsilon q})$, $\mathbf{Q}(\chi)$ has exactly one valuation over r. If $\mathbf{Q}(\sqrt{\varepsilon q})=\mathbf{Q}$ then there is one valuation at ∞ and so $m_r(\chi)=2$ by (II)(i) and Theorem 2.15 as in (II)(iv). If $\mathbf{Q}(\sqrt{\varepsilon q}) \neq \mathbf{Q}$ then either $\varepsilon=-1$ or there are exactly two valuations at ∞. Hence $m_r(\chi)=1$ by (II)(i) and Theorem 2.15.

Thus it suffices to consider the case that $\chi(1)=q-\delta$ with $\delta=\pm 1$.

We will consider two cases.

Case A. Either $\mathbf{Q}(\chi) \neq \mathbf{Q}$ or $\chi=\eta_i$ with $\rho^{6i}=1$ or $\chi=\theta_j$ with $\sigma^{6j}=1$.

Case B. $\mathbf{Q}(\chi)=\mathbf{Q}$ and one of the following holds:

$$q \equiv 5 \pmod 8), \quad \chi=\eta_i, \quad \rho^{4i}=1,$$

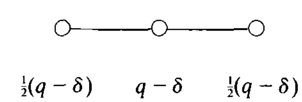
$$q \equiv 3 \pmod 8), \quad \chi=\theta_j, \quad \sigma^{4j}=1.$$

In view of Lemma 6.2 these cases exhaust all possibilities.

Suppose that Case A holds. As $i<\frac{1}{2}(q-1)$ and $j<\frac{1}{2}(q+1)$, $\rho^{2i} \neq 1$ and $\sigma^{2j} \neq 1$. Hence there exists an element x of order $2p^k \not| (q+\delta)$, for some prime p and some natural number k such that either $p=3$ or $\chi(x) \notin \mathbf{Q}$.

Assume first that $p \neq 2$. Let K be the maximal subfield of odd degree in the field of $(q+\delta)$th roots of 1 over $\mathbf{Q}$. By Theorem 2.16 $m_{KQ_l}(\chi)=m_l(\chi)$ for all l. It follows from Table II that $\chi_u(x) \in K(\chi(x))$ for all irreducible characters χ_u of G. Hence Theorem A may be applied. If τ is a root of unity with $\tau^4 \neq 1$ then $2 \not| (\tau+\tau^{-1})$. Thus Theorem A implies that $m_l(\chi)=1$ for $l \neq p, \infty$.

Suppose that $m_p(\chi)=2$. Then χ cannot be irreducible as a Brauer character by Theorem 2.10 and so χ cannot be in a p-block whose Brauer tree is of the form ○——○. Thus χ is in the unique p-block with Brauer tree equal to



Thus $\delta=\varepsilon$ and χ is in the field of p^vth roots of 1 over $\mathbf{Q}$ for some natural number v as the characters of degree $\frac{1}{2}(q-\delta)$ are rational valued on elements whose order divides $q+\delta$. Hence Theorem 2.12 implies that one of the following occurs:

$$\delta=\varepsilon=-1, \quad \chi=\eta_i, \quad \rho^{2ip^k}=1, \quad [\mathbf{Q}_p(\zeta_1):\mathbf{Q}_p]=2,$$

$$\delta=\varepsilon=1, \quad \chi=\theta_j, \quad \sigma^{2jp^k}=1, \quad [\mathbf{Q}_p(\xi_1):\mathbf{Q}_p]=2.$$

Hence $\sqrt{\varepsilon q} \notin \mathbf{Q}_p$ and so q is not a rational square, otherwise $q \equiv 1 \pmod 4$ and $\varepsilon=1$. Thus

$$\left(\frac{-1}{p}\right) = \left(\frac{\varepsilon q}{p}\right) = -1$$

and so $p \equiv 3 \pmod 4$ as in (II)(ii).

Assume now that $p = 2$. Since χ is faithful there exists an element x of order 8 with $\chi(x) = \sqrt{2}$. Since $\chi_u(x) \in \mathbf{Q}_2(\sqrt{2})$ for all irreducible characters χ_u of G it follows from Theorem A that $m_l(\chi) = 1$ for $l \neq 2, \infty$. Furthermore $q \equiv \varepsilon \pmod 8$. Hence $\sqrt{\varepsilon q} \in \mathbf{Q}_2$ and Theorem A implies that $m_l(\chi) = 1$ for $l \neq r, \infty$. Thus $m_l(\chi) = 1$ for $l \neq \infty$.

Suppose finally that Case B holds. Then $\chi(1) = 2d$ with d odd. Let T be a S_2-group of G. Then T is a quaternion group of order 8. Let ψ be the unique irreducible faithful character of T. Then $(\chi, \psi^G) = (\chi_T, \psi) = d$ is odd. Thus $m_l(\chi) = 1$ for $l \neq 2, \infty$ and $m_2(\chi) = 2$ by Theorem 2.1 and the fact that $\chi \in \mathbf{Q}$ as in (II)(iii).

§7. The sporadic simple groups

This section contains the computations of the local Schur indices of all irreducible characters of all the sporadic simple groups and their central extensions. The results are summarized in the tables in Section 8.

Benard [4] has computed the local Schur index of every irreducible character of G where G is a Mathieu group or one of the following:

$$J_1, J_2 = HJ, \qquad J_3 = HJM, \quad HiS, \quad McL.$$

He used the results of Section 2, including Theorem 2.2, together with induce-restrict tables for various subgroups. With two exceptions we are able to recapture his results in a more direct fashion.

The arguments used are pretty much the same for all cases and are illustrated by the example in Section 4. By counting involutions and using Theorem 2.7 and Corollary 2.8 it is possible to determine $m_\infty(\chi)$, though for some of the larger groups a computer is necessary. For instance, F_1 has 194 irreducible characters, and most of these have degree at least 10^{20}. I am indebted to Sidnie M. Feit for some of these computations. Theorem A and the corollaries in Section 3 together with Theorem 2.12 are often enough to determine the local Schur index of the irreducible character χ at all completions of $\mathbf{Q}(\chi)$ with at most one exception. Thus $m_p(\chi)$ is determined for all p by Theorem 2.15. This method yields that $m(\chi) \leqq 2$ in all cases.

The sporadic groups and their covering groups will be treated individually below. In most cases the arguments described in the previous paragraph will not be discussed as they are straightforward. For some groups these arguments suffice to handle all the irreducible characters. For the remaining groups they handle all but a small number of irreducible characters which must then be considered. The following method is very effective.

Suppose that χ, χ_1, χ_2 are irreducible characters of a group with $\chi_u \in \mathbf{Q}(\chi)$ and $m(\chi_u)=1$ for $u=1,2$. Then $m(\chi) \mid (\chi_1\chi_2, \chi)$. Thus if $(\chi_1\chi_2,\chi)$ is odd and $m(\chi) \leqq 2$ it follows that $m(\chi)=1$. The application of this argument requires the use of a computer and I am grateful to J. Neubüser and E. Cleuvers who did the actual computing.

In all cases considered here this method is sufficient to handle all the irreducible characters χ with $m(\chi)=1$. (This contrasts with the character ψ of $\mathrm{Sp}_4(4)$ discussed in Section 5, where this method does not determine $m(\psi)$.) The method is also effective in handling most of the remaining irreducible characters. In fact, a direct use of Theorem 2.2 seems to be necessary in at most four cases. Two of these were handled by Benard [4], another for F_5 is handled below. There remains the case of the faithful irreducible character of the double cover of Suz of degree 228,800, which is still not completely settled.

Throughout the rest of this section G is a sporadic group and $\tilde{G}$ is the universal central extension of G, $Z = Z(\tilde{G})$ is the center of $\tilde{G}$. Irreducible characters of G will usually be denoted by χ or χ_u. If χ is nonprincipal then the kernel Z_0 of χ is contained in Z. It is known that Z is always cyclic. See [10] for a complete description of Z in all cases. Thus if $Z_0 \subseteq Z$ then Z_0 is uniquely determined by $|Z:Z_0|$.

The Mathieu groups. The character tables of the Mathieu groups can, for instance, be found in [13]; those of some of their covering groups can be found in [16]. For the remaining character tables see Tables III and IV. Routine arguments show that $m_p(\chi)=1$ in all cases except if $p=5$ and $\chi=\chi_7$ in the first part of Table IV, when $m_5(\chi_7)=2$.

J_1. $G=\tilde{G}$. The character table of G can be found in [19]. Routine arguments show that $m(\chi)=1$ in all cases.

$J_2 = HJ$. See Section 4.

$J_3 = HJM$. $|Z|=3$. A character table of G can be found in [14] and a table of faithful irreducible characters of $\tilde{G}$ can be found in [20]. It is routine to show that $m(\chi)=1$ unless χ is an irreducible character of G with $\chi(1)=816$. In this case $m_p(\chi)=1$ for $p \neq 2,3$. See Benard [4] for a proof that $m_2(\chi)=m_3(\chi)=2$.

J_4. $G=\tilde{G}$. I am indebted to J. H. Conway who sent me a character table of G. It is routine to show that $m(\chi)=1$ for all χ.

He $= HHM$. $G=\tilde{G}$. I am indebted to J. McKay for a character table of G. It is routine to show that $m(\chi)=1$ for all χ.

HiS. $|Z|=2$. J. S. Frame and independently J. McKay constructed the character table of G. A. Rudvalis constructed the table of faithful irreducible characters of $\tilde{G}$ and computed $m_\infty(\chi)$ in all cases. He also showed that $\mathbf{Q}(i, \chi)$ is a splitting field for every irreducible character χ of $\tilde{G}$. See [22].

There are two faithful irreducible characters χ_1, χ_2 of $\tilde{G}$ with $\chi_1(1)=616$, $\chi_2(1)=924$ and $\mathbf{Q}(\chi_u)=\mathbf{Q}(i)$ for $u=1,2$. It is routine to show that $m(\chi)=1$ for all irreducible characters χ of $\tilde{G}$ except χ_1, χ_2 and those listed in the table of Section 8. The values of $m_p(\chi)$ in the tables of Section 8 are also routine to derive as is the fact that $m_p(\chi_u)=1$ for $u=1,2$ and $p\neq 5$.

There are rational valued characters ψ_1, ψ_2, ψ_3, ψ_4 of degrees 176, 770, 22, 1000 respectively with $m(\psi_u)=1$ for $u=1,\cdots,4$. Direct computation shows that $(\psi_1\psi_2, \chi_1)=3$ and $(\psi_3\psi_4, \chi_2)=1$. Then $m(\chi_u)=1$ for $u=1,2$.

McL. $|Z|=3$. I am indebted to J. McKay for character tables of G and $\tilde{G}$. Routine arguments show that $m(\chi)=1$ for every irreducible character of $\tilde{G}$ except the two listed in the tables in Section 8. If χ_1, χ_2 are these two characters where $\chi_1(1)=3520$ and $\chi_2(1)=4752$, then it is routine to show that $m_p(\chi_1)$ is as in the table of Section 8. It also follows routinely that $m_p(\chi_2)=1$ for $p\neq 3,5$. The fact that $m_5(\chi_2)=1$ and $m_3(\chi_2)=2$ depends on the consideration of subgroups of G. See Benard [4] for a proof.

Suz. $|Z|=6$. A character table of G can be found in [24]. The remaining characters of $\tilde{G}$ were first computed for the Cambridge Atlas. I am indebted to R. Griess for a copy of these. We will consider all the irreducible characters of $\tilde{G}$ with kernel Z_0 as Z_0 ranges over all subgroups of Z. There are four possibilities for Z_0 and these are determined by $|Z:Z_0|$. We will use the notation of character tables which I obtained from Neubüser.

$|Z:Z_0|=1$. There exist irreducible rational valued characters χ_{33}, χ_{40} with $\chi_{33}(1)=100{,}100$ and $\chi_{40}(1)=197{,}120$. Routine arguments show that $m(\chi)=1$ for $\chi\neq\chi_{33},\chi_{40}$. There exist rational valued irreducible characters χ_3, χ_{24} with $\chi_3(1)=364$, $\chi_{24}(1)=64{,}064$ and $(\chi_{33},\chi_3\chi_{64})$ odd. Thus $m(\chi_{33})=1$.

Routine arguments show that $m_5(\chi_{40})=2$ and $m_p(\chi_{40})=1$ for $p\neq 2,3,5$. There exists a pair of algebraically conjugate irreducible characters χ_{31}, χ_{32} in $\mathbf{Q}(\sqrt{13})$ such that $(\chi_{40},\chi_{31}^2-\chi_{31}\chi_{32})=1$. Hence (χ_{40},χ_{31}^2) or $(\chi_{40},\chi_{31}\chi_{32})$ is odd. Thus $\mathbf{Q}(\sqrt{13})$, and hence $\mathbf{Q}_3$, is a splitting field for χ_{40} and so $m_3(\chi_{40})=1$. Hence $m_2(\chi_{40})=2$ by Theorem 2.15.

$|Z:Z_0|=6$. There exists an irreducible character ψ with $\psi(1)=144{,}144$, $\psi(x)=-432$ for an involution x and $\mathbf{Q}(\psi)=\mathbf{Q}(\sqrt{-3})$. Routine arguments show that all faithful irreducible characters of $\tilde{G}$ other than ψ have Schur index 1. There exists an irreducible character θ in $\mathbf{Q}(\sqrt{-3})$ with $\theta(1)=112{,}320$ such that Z is in the kernel of $\psi\theta$ and $(\psi\theta, \psi\theta)$ is odd. Thus $(\psi, \bar{\theta}\chi) = (\psi\theta, \chi)$ is odd for some rational valued irreducible character χ of $G = \tilde{G}/Z$. As $\mathbf{Q}(\sqrt{-3})$ is a splitting field for every rational valued irreducible character of G it follows that $m(\psi) = 1$.

$|Z : Z_0| = 3$. Routine arguments show that all Schur indices are 1.

$|Z : Z_0| = 2$. There exist rational valued characters χ_u as follows:

u	52	53	64	65	66	68	69	74
$\chi_u(1)$	20020	20020	80080	80080	100100	128128	137280	228800.

Routine arguments show that $m_p(\chi_u)$ is as in the table in Section 8 for all p and all χ_v not on the list above. The following computations were performed by J. Neubüser and E. Cleuvers.

There exist rational valued characters χ_2, χ_{49} with $\chi_2(1) = 143$, $\chi_{49}(1) = 4928$, $m(\chi_2) = 1$ and $(\chi_2\chi_{49}, \chi_{68}) = 1$. Thus $m_p(\chi_{68}) = m_p(\chi_{49})$ for all p.

There exist rational valued characters χ_{17}, χ_{75} with $\chi_{17}(1) = 25,025$, $\chi_{75}(1) = 277,200$, $m(\chi_{17}) = 1$ and $(\chi_{75}\chi_{17}, \chi_u)$ odd for $u = 64, 66, 69$. Thus $m_p(\chi_u) = m_p(\chi_{75})$ for all p and $u = 64, 66, 69$.

$(\chi_{75}\chi_u, \chi_{75}\chi_u)$ is odd and $(\chi_{75}\chi_u, \chi_{40})$ is even for $u = 52, 53, 65$. Thus there exists a rational valued irreducible character χ (depending on u) with $m(\chi) = 1$ and $(\chi_u, \chi_{75}\chi) = (\chi_u\chi_{75}, \chi)$ odd. Hence $m_p(\chi_u) = m_p(\chi_{75})$ for $u = 52, 53, 65$.

Routine arguments show that $m_\infty(\chi_{74}) = 2$ and $m_p(\chi_{74}) = 1$ for $p \neq \infty, 2, 3$. It is not yet settled whether $m_2(\chi_{74}) = 1$ or $m_3(\chi_{74}) = 1$.

Co. 1. $|Z| = 2$. The character table of $\tilde{G} = \text{Co.}\,0$ was first computed by J. H. Conway. There exists a faithful irreducible character χ of $\tilde{G}$ with $\chi(1) = 210,496,000$. Routine arguments show that all other irreducible characters of $\tilde{G}$ have Schur index 1. J. Neubüser has computed that $(\chi, \chi_1\chi_2) = 1$ where χ_1, χ_2 are rational valued irreducible characters with $\chi_1(1) = 24$, $\chi_2(1) = 16,347,825$. Thus also $m(\chi) = 1$.

Co. 2. $|Z| = 1$. I am indebted to J. H. Conway for a character table of G. There is a rational valued irreducible character χ with $\chi(1) = 368,874$. Routine arguments show that all other irreducible characters of G have Schur index 1 and $m_p(\chi) = 1$ for $p \neq 2, 5$. There exists an irreducible character ψ with $\psi(1) = 91,125$ and $\mathbf{Q}(\psi) = \mathbf{Q}(\sqrt{-7})$. As $(\chi, \psi\bar{\psi} - \psi^2) = -1$ either $(\chi, \psi\bar{\psi})$ or (χ, ψ^2) is odd, hence $\mathbf{Q}(\sqrt{-7})$ is a splitting field for χ. As $\sqrt{-7} \in \mathbf{Q}_2$ it follows that $m_2(\chi) = 1$. Hence $m(\chi) = 1$ by Theorem 2.15.

Co. 3. $|Z| = 1$. A character table of G can be found in [9]. It is routine to check that $m(\chi) = 1$ for all χ.

Fi$_{22}$. $|Z| = 6$. A character table of Fi_{22} can be found in [12]. I am indebted to

J. Neubüser and H. Pahlings for character tables of the covering groups of Fi_{22}. That for the double cover was constructed by T. Gabrysch, the remaining character tables were constructed by H. Pahlings. We will use the notation of these authors whenever possible.

$|Z_0| \geqq 3$. There is an irreducible rational valued character χ of $\hat{G} = G/Z_0$ with $\chi(1) = 133,056$. Routine arguments show that all other irreducible characters of $\hat{G}$ have Schur index 1. J. Neubüser has computed that $(\chi, \chi_1\chi_2) = 1$ where χ_1, χ_2 are rational valued irreducible characters with $\chi_1(1) = 78$, $\chi_2(1) = 436,800$. Thus also $m(\chi) = 1$.

$|Z_0| = 2$. There are irreducible characters θ_{26}, θ_{34}, θ_{35} with $\mathbf{Q}(\theta_{26}) = \mathbf{Q}(\sqrt{-3})$ and $\mathbf{Q}(\theta_{34}) = \mathbf{Q}(\theta_{35}) = \mathbf{Q}(\sqrt{-3}, \sqrt{-2})$ such that $\theta_{26}(1) = 608,256$ and θ_{34}, θ_{35} are algebraically conjugate of degree 1,297,296. Routine arguments show that all other irreducible characters have Schur index 1 and these have Schur index at most 2. There is a rational valued character χ_{64} of G with $\chi_{64}(1) = 2,555,904$ such that $(\theta_{26}, \theta_{44}\chi_{64}) = 43,131$ is odd. Thus $m(\theta_{26}) = 1$.

There exist algebraically conjugate characters χ_{33}, χ_{34} of G with $\mathbf{Q}(\chi_{33}) = \mathbf{Q}(\sqrt{-2})$ and an irreducible character θ_1 of $\tilde{G}$ with kernel Z_0 such that $\theta_1(1) = 351$ and $Q(\theta_1) = \mathbf{Q}(\sqrt{-3})$. Furthermore $(\theta_1\chi_{33} - \theta_1\chi_{34}, \theta_{34}) = 1$. Thus either $(\theta_1\chi_{33}, \theta_{34})$ or $(\theta_1\chi_{34}, \theta_{34})$ is odd. Hence $m(\theta_{35}) = m(\theta_{34}) = 1$.

$|Z_0| = 1$. There are irreducible faithful characters θ_9 and θ_{30} with $\theta_9(1) = 123,552$, $\theta_{30}(1) = 2,594,592$ and $\mathbf{Q}(\theta_9) = \mathbf{Q}(\theta_{30}) = \mathbf{Q}(\sqrt{-3})$. It is routine to show that every other faithful irreducible character of $\tilde{G}$ has Schur index 1, and that $m_p(\theta_9) = 1$ for $p \neq 7$ and $m(\theta_{30}) \leqq 2$. It follows directly from Theorem 2.12 that $m_7(\theta_9) = 1$ and hence $m(\theta_9) = 1$. Let θ_{22} denote a faithful irreducible character of $\tilde{G}$ with $\theta_{22}(1) = 1,123,200$. Let χ_{65} be the irreducible character of G with $\chi_{65}(1) = 2,729,376$. Then $\mathbf{Q}(\theta_{22}) = \mathbf{Q}(\theta_{30})$, $\mathbf{Q}(\chi_{65}) = \mathbf{Q}$ and $(\chi_{65}\theta_{22}, \theta_{30}) = 123,277$ is odd. Thus $m(\theta_{30}) = 1$.

Fi_{23}. $|Z| = 1$. The character table of G is due to D. C. Hunt. Routine arguments show that all Schur indices are 1.

Fi'_{24}. $|Z| = 3$. The character table of Fi_{24} was computed by D. C. Hunt. The faithful rational irreducible characters of $\tilde{Fi}_{24}$ were computed by T. Gabrysch. Using these results H. Pahlings was able to compute the character table of $\tilde{Fi}_{24}$ and I am indebted to him for sending me a copy.

$|Z_0| = 1$. Routine arguments show that all Schur indices are 1.

$|Z_0| = 3$. There exist algebraically conjugate irreducible characters χ_{97}, χ_{98} in $\mathbf{Q}(\sqrt{13})$ with $\chi_{97}(1) = 77,007,684,600$. Routine arguments show that if $\chi \neq \chi_{97}, \chi_{98}$ is an irreducible character of G then $m(\chi) = 1$. There exist algebraically conjugate irreducible characters χ_{79}, χ_{80} in $\mathbf{Q}(\sqrt{13})$ with $\chi_{79}(1) = 197,813,862,400$

and $(\chi_{79}\chi_{97}, \chi_{79}-\chi_{80}) = -1$. Therefore $(\chi_{97}, \chi_{79}\chi_u) = (\chi_{97}\chi_{79}, \chi_u)$ is odd for $u = 79$ or 80. Thus $\mathbf{Q}(\sqrt{13})$ is a splitting field for χ_{97} and so $m(\chi_{97}) = m(\chi_{98}) = 1$.

LyS. $|Z| = 1$. A character table can be found in [18]. Routine arguments show that all Schur indices are 1.

Ru. $|Z| = 2$. R. Lyons and I had computed the character table of G and part of the character table of $\tilde{G}$. P. Fong computed the character table of $\tilde{G}$ and we will use his notation.

The group G has a 2-block of defect 1. Using this and routine arguments it follows that $m(\chi) = 1$ for every irreducible character χ of G except for χ_{36}, where $\chi_{36}(1) = 105,560$. There exist rational valued characters χ_9, χ_{35} with $\chi_9(1) = 75,400$, $\chi_{35}(1) = 52,780$ such that $(\chi_9\chi_{35}, \chi_{36}) = 1$. Hence $m(\chi_{36}) = 1$.

There are rational valued faithful irreducible characters χ_u of $\tilde{G}$ for $u = 42, 43, 52$ with $\chi_{42}(1) = 250,560$, $\chi_{47}(1) = 48,256$, $\chi_{52}(1) = 87,696$. Routine arguments show that $m_p(\chi_v)$ is as in the table in Section 8 for all p and all $v \neq 42, 43, 52$. Furthermore, $m_\infty(\chi_u) = 2$ and $m_p(\chi_u) = 1$ for $u \neq \infty, 2, 3, 5$. Also $m_3(\chi_{42}) = m_3(\chi_{52}) = 1$.

There exists an irreducible character χ_{44} with $\mathbf{Q}(\chi_{44}) = \mathbf{Q}(i)$ and $\chi_{44}(1) = 28$ such that $(\chi_{44}\chi_{36}, \chi_u)$ is odd for $u = 42, 43, 52$. Hence $\mathbf{Q}(i)$, and thus $\mathbf{Q}_5$, is a splitting field of each χ_u. Therefore $m_5(\chi_u) = 1$ for $u = 42, 43, 52$. Consequently $m_p(\chi_u) = 1$ for $p \neq 2, \infty$ and so $m_2(\chi_u) = 2$ for $u = 42, 52$.

There exists a character χ_{59} with $\mathbf{Q}(\chi_{59}) = \mathbf{Q}(\sqrt{-5})$, $\chi_{59}(1) = 7280$ and $(\chi_{59}\chi_{36}, \chi_{43})$ odd. Thus $\mathbf{Q}(\sqrt{-5})$, and hence $\mathbf{Q}_3$, is a splitting field for χ_{43}. Thus $m_3(\chi_{43}) = 1$ and so $m_2(\chi_{43}) = 2$.

Table (a)

u	$\chi_u(1)$
112	1711786747207680
156	7679219111313750
160	7863520369985280
167	9558000449712000
183	14220776132062500
195	4322693806080
196	10177847623680
221	1566852857856000
240	8740741152000000
227	3824419701473280
225	2995626807613440
241	12604975138529280
245	24515659148820480

Table (b)

u	$\chi_u(1)$
2	4371
3	96255
4	1139374
147	5257393731060471
180	13046344927150080
182	14014628339712000
197	17069098618880
244	24089453696000000

$OS = O'Nan$. $|Z| = 3$. A character table for $\tilde{G}$ can be found in [21]. Routine arguments show that all Schur indices are 1.

$F_2 = BM$. $|Z| = 2$. A table of rational characters of G and $\tilde{G}$ was first constructed by D. C. Hunt and I am indebted to him for a copy of this. Using this table, H. Pahlings constructed the character table of $\tilde{G}$. In the notation of H. Neubüser there exist rational valued irreducible characters χ_u as in table (a). The first five are irreducible characters of G, the last eight are faithful. Routine arguments show that $m(\chi) = 1$ for $\chi \neq \chi_u$ as u ranges over the list in table (a). The characters in table (b) are all rational valued and J. H. Neubüser and E. Cleuvers have shown that $(\chi_u, \chi_v\chi_w)$ is odd as u, v, w range over the following values:

u	v	w
112, 160, 182	2	182
156, 167	2	180
195, 196	3	244
221, 225, 240, 241	2	197
227	2	247
245	4	247

Thus $m(\chi_u) = 1$ for all u.

$F_1 = M = FG$. $|Z| = 1$. The character table of G is due to B. Fischer, D. Livingston and M. Thorne. I am indebted to S. Smith who sent me a copy. Routine arguments show that $m(\chi) = 1$ for all irreducible characters χ.

$F_3 = Th$. $|Z| = 1$. I am indebted to J. G. Thompson for a character table of G. Routine arguments show that $m(\chi) = 1$ for all irreducible characters χ.

$F_5 = Ha$. $|Z| = 1$. A character table of G can be found in [11]. There exists a rational valued character χ with $\chi(1) = 2,661,120$ such that every irreducible character of G distinct from χ has Schur index 1. Furthermore $m_p(\chi) = 1$ for $p \neq 2, 3$. We will show that $m_3(\chi) = 1$ (and hence $m_2(\chi) = 2$) by using detailed information about G. This can all be found in [11] and we will use the notation used there.

A S_2-group of $\mathbf{C}(3_B) = \mathrm{Q}_8$ is a quaternion group of order 8; 2_B, $4_C \in \mathbf{C}(3_B)$, $2_B \cdot 3_B = 6_C$ and $4_C \cdot 3_B = 12_C$.

Let S be a S_2-group of $\mathbf{N}(\langle 3_B \rangle)$ and let $H = \langle 3_B \rangle S$. Then $|S : Q_8| = 2$ and by [11], lemma 2.8 $S/\langle 2_B \rangle = Z_2 \times Z_2 \times Z_2$. Thus S has exponent 4.

Let φ be the faithful irreducible character of $\mathbf{C}(3_B)/\langle 3_B \rangle \simeq Q_8$. Let θ_1, θ_2 be the extensions of φ to $H/\langle 3_B\rangle \simeq S$. Let $\omega \neq 1$ be a linear character of $\langle 3_B\rangle$ and let $\psi = (\omega\varphi)^H$. One gets the following table of values:

	1	2_B	3_B	6_C
χ	$2^8(10395)$	3.2^8	-54	6
θ_u, $u = 1,2$	2	-2	2	-2
ψ	4	-4	-2	2

Furthermore $\chi\theta_u$, $\chi\psi$ vanishes on elements of H other than $1, 2_B, 3_B$ or 6_C. Thus

$$\sum_{x \in H} \chi(x)\theta_u(x^{-1}) \equiv 4(-54-6) \equiv -16.15 \pmod{2^8},$$

$$\sum_{x \in H} \chi(x)\psi(x^{-1}) \equiv 4(54+6) \equiv 16.15 \pmod{2^8}.$$

Therefore (χ, θ_u) and (χ, ψ) are odd.

If θ_u is real valued then $m_\infty(\chi) = m_\infty(\theta_u) = 2$ which is not the case. Therefore $\mathbf{Q}(\theta_u) = \mathbf{Q}(i)$ as S has exponent 4. Thus $[\mathbf{Q}_3(\theta_u) : \mathbf{Q}_3] = 2$ and so $m_3(\chi) = m_3(\psi) = 2$ by Theorem 2.12.

§8. Schur indices of the sporadic groups

G is a sporadic simple group. $\tilde{G}$ is the universal covering group of G. It is known that $Z = Z(\tilde{G})$ is always cyclic [10]. Thus Z_0 is uniquely determined by $|Z : Z_0|$.

Table A. $|Z : Z_0| = 1$

G	$\chi(1)$	$\mathbf{Q}(\chi)$	$M(\chi)$
$J_2 = HJ$	336	$\mathbf{Q}$	2, 3
$J_3 = HJM$	816	$\mathbf{Q}$	2, 3
McL	3520	$\mathbf{Q}$	∞, 5
	[$\chi(\tau) = -64$, τ an involution]		
McL	4752	$\mathbf{Q}$	∞, 3
F_5 (Harada)	2,661,120	$\mathbf{Q}$	2, 3
Suz	197,120	$\mathbf{Q}$	2, 5

Table B. $|Z : Z_0| = 4$

M_{22}	176	$\mathbf{Q}(i)$	5

Table C. $|Z : Z_0| = 2$

G	$\chi(1)$	$\mathbf{Q}(\chi)$	$M(\chi)$
M_{12}	32	$\mathbf{Q}$	$\infty, 2$
$J_2 = HJ$	6	$\mathbf{Q}(\sqrt{5})$	∞
	14	$\mathbf{Q}$	$\infty, 2$
	56	$\mathbf{Q}(\sqrt{5})$	∞
	64	$\mathbf{Q}(\sqrt{5})$	∞
	84	$\mathbf{Q}$	$\infty, 3$
	126	$\mathbf{Q}(\sqrt{5})$	∞
	216	$\mathbf{Q}$	$\infty, 7$
	252	$\mathbf{Q}$	$\infty, 3$
	336	$\mathbf{Q}$	$\infty, 2$
	350	$\mathbf{Q}$	$\infty, 2$
	448	$\mathbf{Q}$	$\infty, 2$
HiS	1000	$\mathbf{Q}$	$\infty, 7$
	1792	$\mathbf{Q}$	$\infty, 2$
	1848	$\mathbf{Q}$	$\infty, 3$
	2520	$\mathbf{Q}(\sqrt{5})$	∞
Ru	8192	$\mathbf{Q}(\sqrt{29})$	∞
	34, 944	$\mathbf{Q}$	$\infty, 3$
	48, 256	$\mathbf{Q}$	$\infty, 2$
	87, 696	$\mathbf{Q}$	$\infty, 2$
	221, 184	$\mathbf{Q}$	$\infty, 2$
	250, 560	$\mathbf{Q}$	$\infty, 2$
Suz	220	$\mathbf{Q}$	$\infty, 2$
	4, 928	$\mathbf{Q}$	$\infty, 5$
	20, 020	$\mathbf{Q}$	$\infty, 2$
	20, 020	$\mathbf{Q}$	$\infty, 2$
	35, 100	$\mathbf{Q}(\sqrt{21})$	∞
	60, 060	$\mathbf{Q}(\sqrt{5})$	∞
	61, 236	$\mathbf{Q}(\sqrt{13})$	∞
	78, 872	$\mathbf{Q}(\sqrt{2})$	∞
	80, 080	$\mathbf{Q}$	$\infty, 2$
	80, 080	$\mathbf{Q}$	$\infty, 2$
	100, 100	$\mathbf{Q}$	$\infty, 2$
	102, 400	$\mathbf{Q}$	$\infty, 11$
	128, 128	$\mathbf{Q}$	$\infty, 5$
	137, 280	$\mathbf{Q}$	$\infty, 2$
	144, 144	$\mathbf{Q}(\sqrt{3})$	∞
	192, 192	$\mathbf{Q}$	$\infty, 5$
	197, 120	$\mathbf{Q}$	$\infty, 2$
	228, 800	$\mathbf{Q}$	$\infty, 2$ or 3
	277, 200	$\mathbf{Q}$	$\infty, 2$
	315, 392	$\mathbf{Q}$	$\infty, 5$

Table I. Faithful Characters of $G = 2HJ$
$\bar{x}$ is the image of x in $\bar{G} \simeq HJ$
(only classes on which not all characters vanish are listed)

$\lvert C_{\bar{G}}(\bar{x})\rvert$ $\lvert\langle\bar{x}\rangle\rvert$	$2^7.3^3.5^2.7$ 1	1920 2	1080 3	36 3	96 4	300 5	300 5	50 5	50 5	24 6	7 7	8 8	10 10	10 10	12 12	15 15	15 15
χ_1	6	-2	-3	0	2	$2+2\alpha'$	$2+2\alpha$	$-1+\alpha'$	$-1+\alpha$	1	-1	0	$1+\alpha'$	$1+\alpha$	-1	α'	α
χ_2	6	-2	-3	0	2	$2+2\alpha$	$2+2\alpha'$	$-1+\alpha$	$-1+\alpha'$	1	-1	0	$1+\alpha$	$1+\alpha'$	-1	α	α'
χ_3	14	6	-4	2	2	4	4	-1	-1	0	0	0	1	1	2	1	1
χ_4	50	10	5	2	2	0	0	0	0	1	1	$\sqrt{-2}$	0	0	-1	0	0
χ_5	50	10	5	2	2	0	0	0	0	1	1	$-\sqrt{-2}$	0	0	-1	0	0
χ_6	56	-8	2	2	0	$2+2\alpha'$	$2+2\alpha$	$-1+\alpha'$	$-1+\alpha$	-2	0	0	α	α'	0	α'	α
χ_7	56	-8	2	2	0	$2+2\alpha$	$2+2\alpha'$	$-1+\alpha$	$-1+\alpha'$	-2	0	0	α'	α	0	α	α'
χ_8	64	0	-8	-2	0	$6+4\alpha'$	$6+4\alpha$	$2\alpha'$	2α	0	1	0	0	0	0	α	α'
χ_9	64	0	-8	-2	0	$6+4\alpha$	$6+4\alpha'$	2α	$2\alpha'$	0	1	0	0	0	0	α'	α
χ_{10}	84	4	-15	0	4	-6	-6	-1	-1	1	0	0	-1	-1	1	0	0
χ_{11}	126	-10	-9	0	2	$4+6\alpha'$	$4+6\alpha$	$-2\alpha'$	-2α	-1	0	0	0	0	-1	1	1
χ_{12}	126	-10	-9	0	2	$4+6\alpha$	$4+6\alpha'$	-2α	$-2\alpha'$	-1	0	0	0	0	-1	1	1
χ_{13}	216	24	0	0	0	6	6	1	1	0	-1	0	-1	-1	0	0	0
χ_{14}	252	-20	9	0	4	2	2	2	2	1	0	0	0	0	1	-1	-1
χ_{15}	336	16	-6	0	0	-4	-4	1	1	-2	0	0	1	1	0	-1	-1
χ_{16}	350	-10	-10	2	-6	0	0	0	0	2	0	0	0	0	0	0	0
χ_{17}	448	0	16	-2	0	-2	-2	-2	-2	0	0	0	0	0	0	1	1

$$\alpha = \frac{-1+\sqrt{5}}{2}, \qquad \alpha' = \frac{-1-\sqrt{5}}{2}$$

Table II. Faithful Characters of $SL_2(q)$, $q = r^n$, r an odd prime

$\lvert C_G(x)\rvert$	$q(q^2-1)$	$q(q^2-1)$	$2q$	$2q$	$2q$	$2q$	$(q-1)$	$(q+1)$
$\lvert\langle x\rangle\rvert$	1	2	r	r	$2r$	$2r$	$(q-1)/(s,q-1)$	$(q+1)/(t,q+1)$
x	1	z	c	c'	zc	zc'	a^s	b^t
1	1	1	1	1	1	1	1	1
Γ	q	q	0	0	0	0	1	-1
η_i	$q+1$	$(-1)^i(q+1)$	1	1	$(-1)^i$	$(-1)^i$	$\rho^{is}+\rho^{-is}$	0
θ_j	$q-1$	$(-1)^j(q+1)$	-1	-1	$(-1)^{j+1}$	$(-1)^{j+1}$	0	$-(\sigma^{jt}+\sigma^{-jt})$
ξ_1	$\frac{1}{2}(q+1)$	$\frac{1}{2}\varepsilon(q+1)$	$\frac{1}{2}(1+\sqrt{\varepsilon q})$	$\frac{1}{2}(1-\sqrt{\varepsilon q})$	$\frac{1}{2}\varepsilon(1+\sqrt{\varepsilon q})$	$\frac{1}{2}\varepsilon(1-\sqrt{\varepsilon q})$	$(-1)^s$	0
ξ_2	$\frac{1}{2}(q+1)$	$\frac{1}{2}\varepsilon(q+1)$	$\frac{1}{2}(1-\sqrt{\varepsilon q})$	$\frac{1}{2}(1+\sqrt{\varepsilon q})$	$\frac{1}{2}\varepsilon(1-\sqrt{\varepsilon q})$	$\frac{1}{2}\varepsilon(1+\sqrt{\varepsilon q})$	$(-1)^s$	0
ζ_1	$\frac{1}{2}(q-1)$	$-\frac{1}{2}\varepsilon(q-1)$	$\frac{1}{2}(-1+\sqrt{\varepsilon q})$	$\frac{1}{2}(-1-\sqrt{\varepsilon q})$	$-\frac{1}{2}\varepsilon(-1+\sqrt{\varepsilon q})$	$-\frac{1}{2}\varepsilon(-1-\sqrt{\varepsilon q})$	0	$(-1)^{t+1}$
ζ_2	$\frac{1}{2}(q-1)$	$-\frac{1}{2}\varepsilon(q-1)$	$\frac{1}{2}(-1-\sqrt{\varepsilon q})$	$\frac{1}{2}(-1+\sqrt{\varepsilon q})$	$-\frac{1}{2}\varepsilon(-1-\sqrt{\varepsilon q})$	$-\frac{1}{2}\varepsilon(-1+\sqrt{\varepsilon q})$	0	$(-1)^{t+1}$

$1 \leqq i,\ s \leqq \frac{1}{2}(q-3)$ $\qquad 1 \leqq j,\ t \leqq \frac{1}{2}(q-1)$

$\varepsilon = (-1)^{(q-1)/2}$, hence $q \equiv \varepsilon \pmod 4$.

ρ is a primitive $(q-1)$st root of 1. $\qquad \sigma$ is a primitive $(q+1)$st root of 1.

Table III. Faithful Characters of $G = 2M_{12}$
$\bar{x}$ is the image of x in $\bar{G} \simeq M_{12}$
(only classes on which not all characters vanish are listed)

$\|C_{\bar{G}}(\bar{x})\|$	$2^6.3^3.5.11$	192	54	36	10	6	8	8	10	11	11
$\|\langle\bar{x}\rangle\|$	1	2	3	3	5	6	8	8	10	11	11
χ_1	10	-2	1	-2	0	1	α	$-\alpha$	0	-1	-1
χ_2	10	-2	1	-2	0	1	$-\alpha$	α	0	-1	-1
χ_3	12	4	3	0	2	1	0	0	0	1	1
χ_4	32	0	-4	2	2	0	0	0	0	-1	-1
χ_5	44	4	-1	2	-1	1	0	0	β	0	0
χ_6	44	4	-1	2	-1	1	0	0	$-\beta$	0	0
χ_7	110	-6	2	2	0	0	α	α	0	0	0
χ_8	110	-6	2	2	0	0	$-\alpha$	$-\alpha$	0	0	0
χ_9	120	8	3	0	0	-1	0	0	0	-1	-1
χ_{10}	160	0	-2	-2	0	0	0	0	0	$-\lambda$	$-\bar{\lambda}$
χ_{11}	160	0	-2	-2	0	0	0	0	0	$-\bar{\lambda}$	$-\lambda$

$$\alpha = \sqrt{-2}, \qquad \beta = \sqrt{-5}, \qquad \lambda = \frac{-1+\sqrt{-11}}{2}$$

Table IV. Faithful Characters of $G = 4M_{22}$ and $12M_{22}$
$\bar{x}$ is the image of x in $\bar{G} \simeq M_{22}$
(only classes on which not all characters vanish are listed)

	$\|C_{\bar{G}}(\bar{x})\|$	$2^7.3^2.5.7.11$	36	5	7	7	8	11	11
	$\|\langle\bar{x}\rangle\|$	1	3	5	7	7	8	11	11
$4M_{22}$	χ_1	56	2	1	0	0	2α	1	1
	χ_2	56	2	1	0	0	-2α	1	1
	χ_3	144	0	-1	$-\lambda$	$-\bar{\lambda}$	0	1	1
	χ_4	144	0	-1	$-\bar{\lambda}$	$-\lambda$	0	1	1
	χ_5	160	-2	0	-1	-1	0	$-\mu$	$-\bar{\mu}$
	χ_6	160	-2	0	-1	-1	0	$-\bar{\mu}$	$-\mu$
	χ_7	176	-4	1	1	1	0	0	0
	χ_8	560	2	0	0	0	0	1	1
$12M_{22}$	χ_1	120	0	0	1	1	2α	-1	-1
	χ_2	120	0	0	1	1	-2α	-1	-1
	χ_3	144	0	-1	$-\lambda$	$-\bar{\lambda}$	0	1	1
	χ_4	144	0	-1	$-\bar{\lambda}$	$-\lambda$	0	1	1
	χ_5	336	0	1	0	0	0	$-\mu$	$-\bar{\mu}$
	χ_6	336	0	1	0	0	0	$-\bar{\mu}$	$-\mu$
	χ_7	384	0	-1	-1	-1	0	-1	-1

$$\alpha = \sqrt{2}, \qquad \lambda = \frac{-1+\sqrt{-7}}{2}, \qquad \mu = \frac{-1+\sqrt{-11}}{2}$$

Each of the above tables contains a complete list of all irreducible characters χ of $\tilde{G}$ with kernel Z_0 which have Schur index $m(\chi)>1$. One character from each set of algebraic conjugates is listed. In all cases but one χ is determined by its degree.

In all cases $m(\chi)=2$. Thus if p ranges over all primes and ∞ then

$$M(\chi)=\{p \mid m_p(\chi)\neq 1\}=\{p \mid m_p(\chi)=2\}$$

together with $\mathbf{Q}(\chi)$ determines the corresponding division algebra. Observe that $m_p(\chi)\neq 1$ for at most 2 places of $\mathbf{Q}(\chi)$ in all cases.

The list is complete, though for one character of the double cover of Suz the answer is not complete. In particular it is complete for all the sporadic simple groups.

Some of these results had previously been obtained by M. Benard [4].

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DEPARTMENT OF MATHEMATICS
YALE UNIVERSITY
NEW HAVEN, CT 06520 USA